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# A note on $p$ -Ascent Sequences

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## Abstract

Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes, and Kitaev in [1], who showed that ascent sequences of length  $n$  are in 1-to-1 correspondence with  $(\mathbf{2} + \mathbf{2})$ -free posets of size  $n$ . In this paper, we introduce a generalization of ascent sequences, which we call  $p$ -ascent sequences, where  $p \geq 1$ . A sequence  $(a_1, \dots, a_n)$  of non-negative integers is a  $p$ -ascent sequence if  $a_0 = 0$  and for all  $i \geq 2$ ,  $a_i$  is at most  $p$  plus the number of ascents in  $(a_1, \dots, a_{i-1})$ . Thus, in our terminology, ascent sequences are 1-ascent sequences. We generalize a result of the authors in [9] by enumerating  $p$ -ascent sequences with respect to the number of 0s. We also generalize a result of Dukes, Kitaev, Remmel, and Steingrímsson in [4] by finding the generating function for the number of  $p$ -ascent sequences which have no consecutive repeated elements. Finally, we initiate the study of pattern-avoiding  $p$ -ascent sequences.

## 1 Introduction

### 1.1 Ascent sequences

Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes, and Kitaev in [1], who showed that ascent sequences of length  $n$  are in 1-to-1 correspondence with  $(\mathbf{2} + \mathbf{2})$ -free posets of size  $n$ . Let  $\mathbb{N} = \{0, 1, \dots\}$  denote the natural numbers and  $\mathbb{N}^*$  denote the set of all words over  $\mathbb{N}$ . A sequence  $(a_1, \dots, a_n) \in \mathbb{N}^n$  is an *ascent sequence of length  $n$*  if and only if it satisfies  $a_1 = 0$  and  $a_i \in [0, 1 + \text{asc}(a_1, \dots, a_{i-1})]$  for all  $2 \leq i \leq n$ , where

$$\text{asc}(a_1, \dots, a_i) = |\{j : a_j < a_{j+1}; 1 \leq j < i\}|$$

is the number of ascents in  $(a_1, \dots, a_n)$ . For instance,  $(0, 1, 0, 2, 3, 1, 0, 0, 2)$  is an ascent sequence which has four ascents. We let  $\text{Asc}$  denote the set of all ascent sequences, where we assume that the empty word is also an ascent sequence. For any  $n \geq 1$ , we let  $\text{Asc}_n$  denote the set of all ascent sequences of length  $n$ . If  $a = (a_1, \dots, a_n) \in \text{Asc}_n$ , we let  $|a| = n$  be the length of  $a$ ,  $\sum a = a_1 + \dots + a_n$  equal the sum of the values of  $a$ ,  $|a|_0$  denote the number of occurrences of 0 in  $a$ , and  $\text{last}(a) = a_n$  denote the last letter of  $a$ . We say that  $a = (a_1, \dots, a_n) \in \text{Asc}_n$

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is an *up-down* ascent sequence if  $a_1 < a_2 > a_3 < a_4 > \dots$ . That is,  $a = (a_1, \dots, a_n) \in \text{Asc}_n$  is an up-down ascent sequence if  $a_i < a_{i+1}$  whenever  $i$  is odd, and  $a_i > a_{i+1}$  whenever  $i$  is even. Throughout this paper, we will often identify a sequence  $(a_1, \dots, a_n)$  in  $\mathbb{N}^n$  with the word  $a_1 \dots a_n$ . Thus, instead of writing, say,  $(0, 0, 0)$ , we will simply write  $000$ , or  $0^3$ .

There has been considerable work on ascent sequences in recent years, see, for example, [1, 4, 6, 9]. Ascent sequences are important because they are in bijection with several other interesting combinatorial objects. To be more precise, it follows from the work of [1, 3, 5] that there are natural bijections between  $\text{Asc}_n$  and the following four classes of combinatorial objects:

- (1) The set of  $(\mathbf{2} + \mathbf{2})$ -free posets of size  $n$ . Here we consider two posets to be equal if they are isomorphic, and an unlabeled poset is said to be  $(\mathbf{2} + \mathbf{2})$ -free if it does not contain an induced subposet that is isomorphic to  $(\mathbf{2} + \mathbf{2})$ , the union of two disjoint 2-element chains.  $(\mathbf{2} + \mathbf{2})$ -free posets are known to be in 1-to-1 correspondence with celebrated *interval orders*.
- (2) The set  $M_n$  of upper triangular matrices of non-negative integers such that no row or column contains all zero entries, and the sum of the entries is  $n$ .
- (3) The set  $R_n$  of permutations of  $[n] = \{1, \dots, n\}$ , where in each occurrence of the pattern 231, either the letters corresponding to the 2 and the 3 are nonadjacent, or else the letters corresponding to the 2 and the 1 are nonadjacent in value. Here, a word contains an occurrence of the pattern 231 if it contains a subsequence of length 3 that is order-isomorphic to 231.
- (4) The set  $Mch_n$  of Stoimenow matchings on  $[2n]$ . A *matching* of the set  $[2n] = \{1, 2, \dots, 2n\}$  is a partition of  $[2n]$  into subsets of size 2, each of which is called an *arc*. The smaller number in an arc is its *opener*, and the larger one is its *closer*. A matching is said to be *Stoimenow* if it has no pair of arcs  $\{a < b\}$  and  $\{c < d\}$  that satisfy one (or both) of the following conditions: (a)  $a = c + 1$  and  $b < d$  and (b)  $a < c$  and  $b = d + 1$ . In other words, a Stoimenow matching has no pair of arcs such that one is nested within the other and either the openers or the closers of the two arcs differ by 1.

Remmel [11] showed that there is an interesting connection between the *Genocchi numbers*  $G_{2n}$  and the *median Genocchi numbers*  $H_{2n-1}$  and up-down ascent sequences. In particular, Remmel showed that  $G_{2n}$  is the number of up-down ascent sequences of length  $2n - 1$ ,  $H_{2n-1}$  is the number of up-down ascent sequences of length  $2n - 2$ .

Let  $p_n$  be the number of ascent sequences of length  $n$ . Bousquet-Mélou et al. [1] proved that

$$P(t) = \sum_{n \geq 0} p_n t^n = \sum_{n \geq 0} \prod_{i=1}^n (1 - (1-t)^i).$$

In fact, Bousquet-Mélou et al. [1] studied a more general generating function

$$F(t, u, v) = \sum_{w \in \text{Asc}} t^{|w|} u^{\text{asc}(w)} v^{\text{last}(w)}$$

and found an explicit form for such a generating function. Kitaev and Remmel [9] studied a refined version of this generating function. That is, they found an explicit formula for the generating function

$$G(t, u, v, z, x) := \sum_{w \in \text{Asc}} t^{|w|} u^{\text{asc}(w)} v^{\text{last}(w)} z^{|w|_0} x^{\text{run}(w)},$$

where for any ascent sequence  $w$ ,  $\text{run}(w) = 0$  if  $w = 0^n$  for some  $n$ , and  $\text{run}(w) = r$  if  $w = 0^r x v$ , where  $x$  is a positive integer and  $v$  is a word. Thus  $\text{run}(w)$  keeps track of the initial sequences



The main goal of this paper is to generalize the results of [9] to  $p$ -ascent sequences. That is, let  $Asc(p)$  denote the set of  $p$ -ascent sequences, where, again, we consider the empty word to be a  $p$ -ascent sequence for any  $p \geq 1$ . Thus, the set of ascent sequences  $Asc$  is  $Asc(1)$  in our terminology. First, we shall study the generating functions

$$G^{(p)}(t, u, v, z, x) := \sum_{w \in Asc(p)} t^{|w|} u^{\text{asc}(w)} v^{\text{last}(w)} z^{|w|_0} x^{\text{run}(w)}. \quad (2)$$

We shall find an explicit formula for  $G^{(p)}(t, u, v, z, x)$  for any  $p \geq 1$  (see Section 2) and then we shall use that formula to prove that

$$A^{(p)}(t, z) := \sum_{w \in Asc(p)} t^{|w|} z^{|w|_0} = 1 + \sum_{n \geq 0} \binom{p+n-1}{n} \frac{zt}{(1-zt)^{n+1}} \prod_{i=1}^n (1 - (1-t)^i). \quad (3)$$

Duncan and Steingrímsson [6] introduced the study of pattern avoidance in ascent sequences. We initiate a similar study for  $p$ -ascent sequences. Given a word  $w = w_1 \dots w_n \in \mathbb{N}^*$ , we let  $\text{red}(w)$  denote the word that is obtained from  $w$  by replacing each copy of the  $i$ -th smallest element in  $w$  by  $i - 1$ . For example,  $\text{red}(238543623) = 015321401$ . Then we say that a word  $u = u_1 \dots u_j$  *occurs* in  $w$  if there exist  $1 \leq i_1 < \dots < i_j \leq n$  such that  $\text{red}(w_{i_1} w_{i_2} \dots w_{i_j}) = u$ . We say that  $w$  *avoids*  $u$  if  $u$  does not occur in  $w$ .

For any word  $u \in \mathbb{N}^*$  such that  $\text{red}(u) = u$ , we let  $a_{n,p,u}$  denote the number of  $p$ -ascent sequences  $a$  of length  $n$  avoiding  $u$  and  $r_{n,p,u}$  denote the number of sequences counted by  $a_{n,p,u}$  with no equal consecutive letters. We prove a number of results about  $a_{n,p,u}$  and  $r_{n,p,u}$ . For example, we will show that for all  $p \geq 1$ ,

$$r_{n,p,10} = \binom{p+n-2}{n-1} \text{ and } a_{n,p,10} = \sum_{s=0}^{n-1} \binom{n-1}{s} \binom{p+s-1}{s}.$$

This paper is organized as follows. In Section 2, we shall find an explicit formula for  $G^{(p)}(t, u, v, z, x)$ . Unfortunately, we can not directly set  $u = 1$  in that formula so that in Section 3, we shall find a formula for  $G^{(p)}(t, 1, 1, 1, x)$  via an alternative proof. This formula will also allow us to find an explicit formula for the generating function for the number of primitive  $p$ -ascent sequences. Finally, in Section 4, we shall study  $a_{n,p,u}$  and  $r_{n,p,u}$  for certain patterns  $u$  of lengths 2 and 3.

## 2 Main results

For  $r \geq 1$ , let  $G_r^{(p)}(t, u, v, z)$  denote the coefficient of  $x^r$  in  $G^{(p)}(t, u, v, z, x)$ . Thus  $G_r^{(p)}(t, u, v, z)$  is the generating function of those  $p$ -ascent sequences that begin with  $r \geq 1$  0s followed by some element between 1 and  $p$ . We let  $G_{a,\ell,m,n}^{(p,r)}$  denote the number of  $p$ -ascent sequences of length  $n$ , which begin with  $r$  0s followed by some element between 1 and  $p$ , have  $a$  ascents, last letter  $\ell$ , and a total of  $m$  zeros. We then let

$$G_r^{(p)}(t, u, v, z) = \sum_{a,\ell,m \geq 0, n \geq r+1} G_{a,\ell,m,n}^{(p,r)} t^n u^a v^\ell z^m. \quad (4)$$

The sequences of the form  $0^n$  contribute a term  $1 + tz + (tz)^2 + \dots = \frac{1}{1-tz}$  to  $G_r^{(p)}(t, u, v, z)$  since they have no ascents and no initial run of 0s (by definition). Hence

$$G^{(p)}(t, u, v, x, z) = \frac{1}{1-tz} + \sum_{r \geq 1} x^r G_r^{(p)}(t, u, v, z). \quad (5)$$

**Lemma 1.** *For  $r \geq 1$ , the generating function  $G_r^{(p)}(t, u, v, z)$  satisfies*

$$(v-1-tv(1-u))G_r^{(p)}(t, u, v, z) = t^{r+1}z^r uv(v^p-1) + t((v-1)z-v)G_r^{(p)}(t, u, 1, z) + tuv^{p+1}G_r^{(p)}(t, uv, 1, z). \quad (6)$$

*Proof.* Our proof follows the same steps as the proof of the  $p = 1$  case of the result that was provided in [9]. Fix  $r \geq 1$ . Let  $x' = (x_1, \dots, x_{n-1})$  be an ascent sequence beginning with  $r$  0s followed by a nonzero element, with  $a$  ascents and  $m$  zeros, where  $x_{n-1} = \ell$ . Then  $x = (x_1, \dots, x_{n-1}, i)$  is an ascent sequence if and only if  $i \in [0, a+p]$ . Clearly,  $x$  also begins with  $r$  0s followed by a nonzero element. Now, if  $i = 0$ , the sequence  $x$  has  $a$  ascents and  $m+1$  zeros. If  $1 \leq i \leq \ell$ ,  $x$  has  $a$  ascents and  $m$  zeros. Finally if  $i \in [\ell+1, a+p]$ , then  $x$  has  $a+1$  ascents and  $m$  zeros. Counting the sequences  $0 \dots 0q$  with  $r$  0s and  $1 \leq q \leq p$  separately, we have

$$\begin{aligned} G_r^{(p)}(t, u, v, z) &= t^{r+1}uvz^r \frac{v^p-1}{v-1} + \\ &\quad \sum_{\substack{a, \ell, m \geq 0 \\ n \geq r+1}} G_{a, \ell, m, n}^{(p, r)} t^{n+1} \left( u^a v^0 z^{m+1} + \sum_{i=1}^{\ell} u^a v^i z^m + \sum_{i=\ell+1}^{a+p} u^{a+1} v^i z^m \right) \\ &= t^{r+1}uvz^r \frac{v^p-1}{v-1} + t \sum_{\substack{a, \ell, m \geq 0 \\ n \geq r+1}} G_{a, \ell, m, n}^{(p, r)} t^n u^a z^m \left( z + \frac{v^{\ell+1}-v}{v-1} + u \frac{v^{a+p+1}-v^{\ell+1}}{v-1} \right) \\ &= t^{r+1}uvz^r \frac{v^p-1}{v-1} + tz G_r^{(p)}(t, u, 1, z) + \\ &\quad tv \frac{G_r^{(p)}(t, u, v, z) - G_r^{(p)}(t, u, 1, z)}{v-1} + tuv \frac{v^p G_r^{(p)}(t, uv, 1, z) - G_r^{(p)}(t, u, v, z)}{v-1}. \end{aligned}$$

The result follows.  $\square$

Next, just like in the proof of the  $p = 1$  case in [9], we use the kernel method to proceed. Setting  $(v-1-tv(1-u)) = 0$  and solving for  $v$ , we obtain that the substitution  $v = 1/(1+t(u-1))$  will eliminate the left-hand side of (6). We can then solve for  $G_r^{(p)}(t, u, 1, z)$  to obtain that

$$G_r^{(p)}(t, u, 1, z) = \frac{t^r z^r u}{\gamma_1 \delta_1^p} (1 - \delta_1^p) + \frac{u}{\gamma_1 \delta_1^p} G_r^{(p)}\left(t, \frac{u}{\delta_1}, 1, z\right) \quad (7)$$

where  $\delta_1 = 1 + t(u-1)$  and  $\gamma_1 = 1 + zt(u-1)$ .

Next we let  $\delta_k = u - (1-t)^k(u-1)$  and  $\gamma_k = u - (1-zt)(1-t)^{k-1}(u-1)$  for  $k \geq 1$ . We also set  $\delta_0 = \gamma_0 = 1$ . Observe that  $\delta_1 = u - (1-t)(u-1) = 1 + t(u-1)$  and  $\gamma_1 = u - (1-zt)(u-1) = 1 + zt(u-1)$ .

For any function of  $f(u)$ , we shall write  $f(u)|_{u=\frac{u}{\delta_k}}$  for  $f(u/\delta_k)$ . It is then easy to check that

$$\begin{aligned}\delta_s|_{u=\frac{u}{\delta_k}} &= \frac{\delta_{s+k}}{\delta_k}, \quad \gamma_s|_{u=\frac{u}{\delta_k}} = \frac{\gamma_{s+k}}{\delta_k}, \quad \frac{u}{\delta_s}|_{u=\frac{u}{\delta_k}} = \frac{u}{\delta_{s+k}}, \text{ and} \\ (u-1)|_{u=\frac{u}{\delta_k}} &= \frac{(1-t)^k(u-1)}{\delta_k}.\end{aligned}$$

Using these relations, one can iterate the recursion (7). For example,

$$\begin{aligned}\frac{u^k}{\gamma_1 \cdots \gamma_k \delta_k^p} G_r^{(p)}\left(t, \frac{u}{\delta_k}, 1, z\right) &= \frac{u^k}{\gamma_1 \cdots \gamma_k \delta_k^p} \left( \frac{t^r z^r \frac{u}{\delta_k} \left(1 - \frac{\delta_{k+1}^p}{\delta_k^p}\right)}{\frac{\gamma_{k+1}}{\delta_k} \frac{\delta_{k+1}^p}{\delta_k^p}} + \frac{\frac{u}{\delta_k}}{\frac{\gamma_{k+1}}{\delta_k} \frac{\delta_{k+1}^p}{\delta_k^p}} G_r^{(p)}\left(t, \frac{u}{\delta_{k+1}}, 1, z\right) \right) \\ &= \frac{t^r z^r u^{k+1} \left(1 - \frac{\delta_{k+1}^p}{\delta_k^p}\right)}{\gamma_1 \cdots \gamma_{k+1} \delta_{k+1}^p} + \frac{u^{k+1}}{\gamma_1 \cdots \gamma_{k+1} \delta_{k+1}^p} G_r^{(p)}\left(t, \frac{u}{\delta_{k+1}}, 1, z\right).\end{aligned}$$

Thus, by iterating recursion (7), we can derive that

$$G_r^{(p)}(t, u, 1, z) = \frac{t^r z^r u(1 - \delta_1^p)}{\gamma_1 \delta_1^p} + \sum_{k=2}^{\infty} \frac{t^r z^r u^k \left(1 - \frac{\delta_k^p}{\delta_{k-1}^p}\right)}{\gamma_1 \cdots \gamma_k \delta_k^p}. \quad (8)$$

Note that since  $\delta_0 = 1$ , we can rewrite  $\frac{t^{r+1} z^r u(1 - \delta_1^p)}{\gamma_1 \delta_1^p}$  as  $\frac{t^r z^r u(\delta_0^p - \delta_1^p)}{\gamma_1 \delta_0^p \delta_1^p}$  and we can rewrite  $\frac{t^r z^r u^k \left(1 - \frac{\delta_k^p}{\delta_{k-1}^p}\right)}{\gamma_1 \cdots \gamma_k \delta_k^p}$  as  $\frac{t^r z^r u(\delta_{k-1}^p - \delta_k^p)}{\gamma_1 \cdots \gamma_k \delta_{k-1}^p \delta_k^p}$ . Thus we have proved the following theorem.

**Theorem 2.**

$$G_r^{(p)}(t, u, 1, z) = \sum_{k=1}^{\infty} \frac{t^r z^r u^k (\delta_{k-1}^p - \delta_k^p)}{\gamma_1 \cdots \gamma_k \delta_{k-1}^p \delta_k^p}. \quad (9)$$

Note that we can rewrite (6) as

$$G_r^{(p)}(t, u, v, z) = \frac{t^{r+1} z^r u v (v^p - 1)}{v \delta_1 - 1} + \frac{t(z(v-1) - v)}{v \delta_1 - 1} G_r^{(p)}(t, u, 1, z) + \frac{u v^{p+1} t}{v \delta_1 - 1} G_r^{(p)}(t, u v, 1, z). \quad (10)$$

For  $s \geq 1$ , we let

$$\bar{\delta}_s = \delta_s|_{u=uv} = uv - (1-t)^s(uv-1) \text{ and } \bar{\gamma}_s = \gamma_s|_{u=uv} = uv - (1-zt)(1-t)^{s-1}(uv-1)$$

and set  $\bar{\delta}_0 = \bar{\gamma}_0 = 1$ . Then using (10) and (9), we have the following theorem.

**Theorem 3.** For all  $r \geq 1$ ,

$$\begin{aligned}G_r^{(p)}(t, u, v, z) &= \\ t^r z^r &\left( \frac{t u v (v^p - 1)}{v \delta_1 - 1} + \frac{t(z(v-1) - v)}{v \delta_1 - 1} \sum_{k \geq 1} \frac{(\delta_{k-1}^p - \delta_k^p)}{\gamma_1 \cdots \gamma_k \delta_{k-1}^p \delta_k^p} + \frac{t u v^{p+1}}{v \delta_1 - 1} \sum_{k \geq 1} \frac{(\bar{\delta}_{k-1}^p - \bar{\delta}_k^p)}{\bar{\gamma}_1 \cdots \bar{\gamma}_k \bar{\delta}_{k-1}^p \bar{\delta}_k^p} \right). \quad (11)\end{aligned}$$

It is easy to see from Theorem 3 that  $G_r^{(p)}(t, u, v, z) = t^{r-1}z^{r-1}G_1^{(p)}(t, u, v, z)$ . This is also easy to see combinatorially since every ascent sequence counted by  $G_r^{(p)}(t, u, v, z)$  is of the form  $0^{r-1}a$ , where  $a$  is a  $p$ -ascent sequence counted by  $G_1^{(p)}(t, u, v, z)$ . Hence

$$\begin{aligned} G^{(p)}(t, u, v, z, x) &= \frac{1}{1-tz} + \sum_{r \geq 1} G_r^{(p)}(t, u, v, z) x^r = \frac{1}{1-tz} + \sum_{r \geq 1} t^{r-1}z^{r-1}G_1^{(p)}(t, u, v, z) x^r \\ &= \frac{1}{1-tz} + \frac{x}{1-tzx} G_1^{(p)}(t, u, v, z). \end{aligned}$$

Thus we have the following theorem.

**Theorem 4.**  $G^{(p)}(t, u, v, z, x) = \frac{1}{1-tz} + \frac{x}{1-tzx} G_1^{(p)}(t, u, v, z)$ .

### 3 Specializations of our general results

In this section, we shall compute the generating function for  $p$ -ascent sequences by length and the number of zeros.

For  $n \geq 1$ , let  $H_{a,b,\ell,n}^{(p)}$  denote the number of  $p$ -ascent sequences of length  $n$  with  $a$  ascents and  $b$  zeros which have last letter  $\ell$ . Then we first wish to compute

$$H^{(p)}(t, u, v, z) = \sum_{n \geq 1, a, b, \ell \geq 0} H_{a,b,\ell,n}^{(p)} u^a z^b v^\ell t^n. \quad (12)$$

Using the same reasoning as in the previous section, we see that

$$\begin{aligned} H^{(p)}(t, u, v, z) &= tz + \sum_{\substack{a, b, \ell \geq 0 \\ n \geq 1}} H_{a,b,\ell,n}^{(p)} t^{n+1} \left( u^a v^0 z^{b+1} + \sum_{i=1}^{\ell} u^a v^i z^b + \sum_{i=\ell+1}^{a+p} u^{a+1} v^i z^b \right) \\ &= tz + t \sum_{\substack{a, b, \ell \geq 0 \\ n \geq r+1}} H_{a,b,\ell,n}^{(p)} t^n u^a z^b \left( z + \frac{v^{\ell+1} - v}{v-1} + u \frac{v^{a+p+1} - v^{\ell+1}}{v-1} \right) \\ &= tz + tz H^{(p)}(t, u, 1, z) + \frac{tv}{v-1} \left( H^{(p)}(t, u, v, z) - H^{(p)}(t, u, 1, z) \right) + \\ &\quad \frac{tuv}{v-1} \left( H^{(p)}(t, uv, 1, z) - H^{(p)}(t, u, v, z) \right). \end{aligned}$$

Solving for  $H^{(p)}(t, u, v, z)$ , we see that we have the following lemma.

**Lemma 5.**

$$\begin{aligned} (v\delta_1 - 1)H^{(p)}(t, u, v, z) &= \\ (v-1)tz + t(z(v-1) - v)H^{(p)}(t, u, 1, z) + tuv^{p+1}H^{(p)}(t, uv, 1, z). \end{aligned} \quad (13)$$

Again, the substitution  $v = \frac{1}{\delta_1}$  eliminates the left-hand side of (13). We can then solve for  $H^{(p)}(u, 1, z, t)$  to obtain the recursion

$$H^{(p)}(t, u, 1, z) = \frac{(1 - \delta_1)z}{\gamma_1} + \frac{u}{\gamma_1 \delta_1^p} H^{(p)}\left(t, \frac{u}{\delta_1}, 1, z\right). \quad (14)$$



We can iterate the recursion (14) in the same manner as we iterated the recursion (7) in the previous section to prove that

$$H^{(p)}(t, u, 1, z) = \sum_{n \geq 0} \frac{(\delta_n - \delta_{n+1})zu^n}{\gamma_1 \cdots \gamma_{n+1}\delta_n^p}. \quad (15)$$

We can easily check that for all  $n \geq 0$ ,  $\delta_n - \delta_{n+1} = (1-u)t(1-t)^n$ . Thus, as a power series in  $u$ , we can conclude the following.

**Theorem 6.**  $H^{(p)}(t, u, 1, z) = \sum_{n=0}^{\infty} \frac{zt(1-u)u^n(1-t)^n}{\delta_n^p \prod_{i=1}^{n+1} \gamma_i}.$

We would like to set  $u = 1$  in the power series  $\sum_{s=0}^{\infty} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1} \gamma_i}$ , but the factor  $(1-u)$  in the series does not allow us to do that in this form. Thus our next step is to rewrite the series in a form where it is obvious that we can set  $u = 1$  in the series. To that end, observe that for  $k \geq 1$ ,  $\delta_k = u - (1-t)^k(u-1) = 1 + u - 1 - (1-t)^k(u-1) = 1 - ((1-t)^k - 1)(u-1)$ , so that by Newton's binomial theorem,

$$\begin{aligned} \frac{1}{\delta_k^p} &= \left( \frac{1}{1 - (u-1)((1-t)^k - 1)} \right)^p = \sum_{n=0}^{\infty} \binom{p-1+n}{n} ((u-1)((1-t)^k - 1))^n \\ &= \sum_{n=0}^{\infty} \binom{p-1+n}{n} (u-1)^n \left( \sum_{m=0}^n (-1)^{n-m} \binom{n}{m} (1-t)^{km} \right). \end{aligned} \quad (16)$$

Substituting (16) into Theorem 6, we see that

$$\begin{aligned} H^{(p)}(t, u, 1, z) &= \frac{zt(1-u)}{\gamma_1} + \sum_{k \geq 1} \frac{zt(1-u)u^k(1-t)^k}{\prod_{i=1}^{k+1} \gamma_i} \sum_{n \geq 0} \binom{p-1+n}{n} (u-1)^n \sum_{m=0}^n (-1)^{n-m} \binom{n}{m} (1-t)^{km} = \\ &= \frac{zt(1-u)}{\gamma_1} + \sum_{n \geq 0} \sum_{m=0}^n (-1)^{n-m-1} \binom{n}{m} (u-1)^{n-m} zt \sum_{k \geq 1} \frac{(u-1)^{m+1} u^k (1-t)^{k(m+1)}}{\prod_{i=1}^{k+1} \gamma_i} = \\ &= \frac{zt(1-u)}{\gamma_1} + \sum_{n \geq 0} \binom{p-1+n}{n} \sum_{m=0}^n (-1)^{n-m-1} \binom{n}{m} (u-1)^{n-m} \frac{zt}{(1-zt)^{m+1}} \times \\ &\quad \sum_{k \geq 1} \frac{(u-1)^{m+1} (1-zt)^{m+1} u^k (1-t)^{k(m+1)}}{\prod_{i=1}^{k+1} \gamma_i}. \end{aligned}$$

In [9], we have proved the following lemma.

**Lemma 7.**

$$\sum_{k \geq 0} \frac{(u-1)^{m+1} (1-zt)^{m+1} u^k (1-t)^{k(m+1)}}{\prod_{i=1}^{k+1} \gamma_i} = - \sum_{j=0}^m (u-1)^j (1-zt)^j u^{m-j} \prod_{i=j+1}^m (1 - ((1-t)^i)).$$

It thus follows that

$$H^{(p)}(t, u, 1, z) = \frac{zt(1-u)}{\gamma_1} + \sum_{n \geq 0} \binom{p-1+n}{n} \sum_{m=0}^n (-1)^{n-m-1} \binom{n}{m} (u-1)^{n-m} \frac{zt}{(1-zt)^{m+1}} \times \\ \left( -\frac{(u-1)^{m+1}(1-zt)^{m+1}}{\gamma_1} - \sum_{j=0}^m (u-1)^j (1-zt)^j u^{m-j} \prod_{i=j+1}^m (1-(1-t)^i) \right).$$

There is no problem in setting  $u = 1$  in this expression to obtain that

$$H^{(p)}(t, 1, 1, z) = \sum_{n \geq 0} \binom{p-1+n}{n} \frac{zt}{(1-zt)^{n+1}} \prod_{i=1}^n (1-(1-t)^i). \quad (17)$$

Clearly, our definitions ensure that  $1 + H(t, 1, 1, z) = A^{(p)}(t, z)$  as defined in the introduction so that we have the following theorem.

**Theorem 8.** *For all  $p \geq 1$ ,*

$$A^{(p)}(t, z) = \sum_{w \in \text{Asc}(p)} t^{|w|} z^{|w|_0} = 1 + \sum_{n \geq 0} \binom{p-1+n}{n} \frac{zt}{(1-zt)^{n+1}} \prod_{i=1}^n (1-(1-t)^i). \quad (18)$$

The case  $p = 1$  in Theorem 8 gives exactly the same formula for  $A^{(1)}(t, z)$  as that derived in [9], which should be the case. We also note that the authors conjectured in [9] that

$$1 + \sum_{k=0}^{\infty} \frac{zt}{(1-zt)^{k+1}} \prod_{i=1}^k (1-((1-t)^i)) = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^m (1-(1-t)^{i-1}(1-zt)). \quad (19)$$

This was proved independently by Jelínek [7], Levande [10], and Yan [13]. It would be interesting to find an analogue of this relation for  $p > 1$ .

Next we can use the same techniques as in [4] to find the generating function for the number of *primitive  $p$ -ascent sequences*. That is, let  $r_{n,p}$  denote the number of  $p$ -ascent sequences  $a$  of length  $n$  such that  $a$  has no consecutive repeated letters and  $a_{n,p}$  denote the number of  $p$ -ascent sequences  $a$  of length  $n$ .

If  $R^{(p)}(t) = 1 + \sum_{n \geq 1} r_{n,p} t^n$  and  $A^{(p)}(t) = 1 + \sum_{n \geq 1} a_{n,p} t^n$ , then it is easy to see that

$$A^{(p)}(t) = A^{(p)}(t, 1) = R^{(p)}\left(\frac{t}{1-t}\right) = R^{(p)}(t + t^2 + \dots), \quad (20)$$

since each element in a primitive  $p$ -ascent sequence can be repeated any specified number of times. Setting  $x = \frac{t}{1-t}$  so that  $t = \frac{x}{1+x}$ , we see that (20) implies that

$$R^{(p)}(x) = A^{(p)}\left(\frac{x}{1+x}\right). \quad (21)$$

Using our formula (18) for  $A^{(p)}(t)$  and simplifying will yield the following theorem.

**Theorem 9.** *For all  $p \geq 1$ ,  $R^{(p)}(t) = 1 + t \sum_{n=0}^{\infty} \binom{p-1+n}{n} (1+t)^n \prod_{i=1}^n \left(1 - \left(\frac{1}{1+t}\right)^i\right)$ .*

Finally if we replace  $t$  by  $t + t^2 + \dots + t^k = t \frac{(t^k - 1)}{t - 1}$  in Theorem 9, then we can obtain the generating function for the number of  $p$ -ascent sequences  $a$  such that the maximum length of a consecutive sequence of repeated letters is less than or equal to  $k$ :

$$1 + t \frac{t^k - 1}{t - 1} \sum_{n=0}^{\infty} \binom{p - 1 + n}{n} \left( \frac{t^{k+1} - 1}{t - 1} \right)^n \prod_{i=1}^n \left( 1 - \left( \frac{t - 1}{t^{k+1} - 1} \right)^i \right). \quad (22)$$

## 4 Pattern avoidance in $p$ -ascent sequences

In this section, we shall prove some simple results about pattern avoidance in  $p$ -ascent sequences thus extending the studies initiated in [6] for ascent sequences.

We begin by considering patterns of length 2. There are three such patterns, 00, 01, and 10. Recall that  $a_{n,p,u}$  (resp.,  $r_{n,p,u}$ ) is the number of (resp., primitive)  $p$ -ascent sequences of length  $n$  that avoid a pattern  $u$ . The only  $p$ -ascent sequences that avoid 01 are the sequences that consist of all zeros so that  $a_{n,p,01} = 1$  for all  $n, p \geq 1$  and  $r_{n,p,01}$  equals 1 if  $n = 1$  and 0 otherwise.

### 10-avoiding $p$ -ascent sequences

Let us consider  $r_{n,p,10}$ . In this case, we are looking for  $p$ -ascent sequences which avoid 10 and have no repeated letters. It is clear that any such a sequence  $a$  must be of the form  $a = a_1 \dots a_n$ , where  $0 = a_1 < a_2 < \dots < a_n$ . For each  $1 \leq i \leq n$ , the word  $a_1 \dots a_i$  has  $i - 1$  ascents so that  $a_{i+1} \leq i - 1 + p$ . It follows that  $r_{n,p,10}$  counts all words  $a_1 a_2 \dots a_n$ , where  $0 = a_1 < a_2 < \dots < a_n \leq p + n - 2$  so that  $r_{n,p,10} = \binom{p + n - 2}{n - 1}$ . Hence by Newton's Binomial Theorem,

$$R_{10}^{(p)}(t) = 1 + \sum_{n \geq 1} \binom{p - 1 + n - 1}{n - 1} t^n = 1 + \frac{t}{(1 - t)^p}. \quad (23)$$

It is easy to see that the  $p$ -ascent sequences counted by  $a_{n,p,10}$  arise by taking a sequence  $d_1 \dots d_s$  counted by  $r_{s,p,10}$  for some  $s \leq n$  and replacing each letter  $d_i$  by one or more copies so that the resulting word is of length  $n$ . The number of ways to do this for a given  $d_1 \dots d_s$  is the number of solutions to  $b_1 + \dots + b_s = n$ , where  $b_i \geq 1$ , which is  $\binom{n-1}{s-1}$ . Thus

$$a_{n,p,10} = \sum_{s=1}^n \binom{n-1}{s} r_{s,p,10} = \sum_{s=1}^n \binom{n-1}{s-1} \binom{p+s-2}{s-1} = \sum_{s=0}^{n-1} \binom{n-1}{s} \binom{p+s-1}{s}. \quad (24)$$

It also follows that  $A_{10}^{(p)}(t) = R_{10}^{(p)}\left(\frac{t}{1-t}\right) = 1 + \frac{t(1-t)^{p-1}}{(1-2t)^p}$ .

We note that the sequence  $(a_{n,2,10})_{n \geq 1}$  starts out 1, 3, 8, 20, 48, 112, 256, ... and this is the sequence A001792 in the OEIS [12] which has many combinatorial interpretations.

### 00-avoiding $p$ -ascent sequences

If a  $p$ -ascent sequence  $a = a_1 \dots a_n$  avoids 00, then all its elements must be distinct. Note that for each  $2 \leq i \leq n$ ,  $a_1 \dots a_{i-1}$  can have at most  $i - 2$  ascents so that  $a_i \leq p + i - 2$ . Let  $\max(a)$  denote the maximum of  $\{a_1, \dots, a_n\}$ . If  $a$  avoids 00, then by the pigeon hole principle, it must be the case that  $\max(a) \geq n - 1$ . Thus, if  $a$  avoids 00, then  $n - 1 \leq \max(a) \leq n + p - 2$ .

Now consider 2-ascent sequences that avoid 00. Suppose that  $a = a_1 \dots a_n$  is a 2-ascent sequence which avoids 00. Then we know that  $\max(a) \in \{n-1, n\}$ . If  $\max(a) = n$ ,  $a$  must be strictly increasing and there must be some smallest  $k \geq 1$  such that  $a_k = k$ . In such a situation, it is easy to see that  $a$  must be of the form  $0, 1, \dots, k-2, k, k+1, \dots, n$ . Thus there are  $n-1$  2-ascent sequences  $a$  of length  $n$  such that  $a$  avoids 00 and  $\max(a) = n$ .

Next, suppose that  $a = a_1 \dots a_n$  is a 2-ascent sequence that avoids 00 and  $\max(a) = n-1$ . Then there are two cases. Namely, it could be that there is no  $k \leq n$  such that  $a_k = k$ . In that case,  $a$  is the increasing sequence  $a = 012 \dots (n-1)$ . Otherwise, let  $j$  equal the smallest  $i$  such that  $a_i = i$ . Then  $a$  must be strictly increasing up to  $a_j$  so that  $a$  starts out  $012 \dots (j-2)j$ . Since  $\max(a) = n-1$ , it follows that  $\{a_1, \dots, a_n\} = \{0, 1, \dots, n-1\}$  so that there must be some  $j < k \leq n$  such that  $a_k = j-1$ . In that case,  $a_{k-1} > a_k$  so that  $a$  has at least one descent. However, if  $\max(a) = n-1$ ,  $a$  can have at most one descent. Thus, once we have placed  $j-1$ , the remaining elements must be placed in increasing order. It is then easy to check that no matter where we place  $j-1$  after position  $j$ , the resulting sequence will be a 2-ascent sequence. It follows that the number of 2-ascent sequences which avoid 00 and have one descent is  $\sum_{j=1}^{n-1} (n-j) = \binom{n-1}{2}$ .

Thus, we have the following theorem.

**Theorem 10.** For all  $n \geq 1$ ,  $a_{n,2,00} = n-1 + 1 + \binom{n-1}{2} = 1 + \binom{n}{2}$ .

The sequence  $(a_{n,3,00})_{n \geq 1}$  starts out  $1, 3, 9, 24, 57, 122, 239, 435, 745, 1213, 1893, 2850, \dots$ , which is the sequence A089830 in the OEIS [12], whose generating function is  $\frac{1-3x+6x^2-5x^3+3x^4-x^5}{(1-x)^6}$ .

In this case, if  $a = a_1 \dots a_n$  is a 3-ascent sequence which avoids 00, then we know that  $n-1 \leq \max(a) \leq n+1$ . We shall prove that

$$\sum_{n \geq 1} a_{n,3,00} x^n = \frac{x(1-3x+6x^2-5x^3+3x^4-x^5)}{(1-x)^6}$$

by classifying the 3-ascent sequences  $a$  which avoid 00 by the  $\max(a)$  and  $\text{des}(a)$ , where  $\text{des}(a)$  is the number of *descents* in  $a$ , that is, the number of elements followed by smaller elements.

**Case 1.**  $\text{des}(a) = 0$ .

Suppose that  $a = a_1 \dots a_n$  is an increasing 3-ascent sequence that avoids 00. Now, if  $\max(a) = n-1$ , then  $a = 012 \dots (n-1)$ . If  $\max(a) = n$ , then exactly one element from  $[n] = \{1, \dots, n-1\}$  does not appear in  $a$ . If  $i$  does not appear in  $a$ , then  $a = 01 \dots (i-1)(i+1)(i+2) \dots n$ , which is a 3-ascent sequence. Thus, there are  $n-1$  increasing 3-ascent sequences whose maximum is  $n$ . Finally, if  $\max(a) = n+1$ , then two elements from  $[n]$  do not appear in  $a$ . Again, it is easy to check that no matter which two elements from  $[n]$  we leave out, the resulting increasing sequence will be a 3-ascent sequence. Thus, there are  $\binom{n}{2}$  increasing 3-ascent sequences whose maximum is  $n+1$ . Therefore, the total number of increasing 3-ascent sequences of length  $n$  is  $1 + (n-1) + \binom{n}{2} = \binom{n+1}{2}$ .

**Case 2.**  $\text{des}(a) = 1$ .

In this case, if  $a = a_1 \dots a_n$  is a 3-ascent sequence such that  $\text{des}(a) = 1$  and  $a$  avoids 00, then  $\max(a) \in \{n-1, n\}$ . Suppose that  $a_j > a_{j-1}$ . Then we have two subcases depending on whether  $a_j = j$  or  $a_j = j+1$ .

If  $a_j = j + 1$ , then there must be two elements  $1 \leq u < v \leq j$ , which do not appear in  $a_1 \dots a_j$ . Clearly, we have  $\binom{j}{2}$  ways to pick  $u$  and  $v$ . We then have three subcases depending on whether  $u$  and  $v$  appear in  $a$ . If both  $u$  and  $v$  appear in  $a$ , then  $a$  must start out  $a_1 \dots a_j uv$  so that  $a_{j+3} \dots a_n$  must be an increasing sequence from  $[n] - [j+1]$  of length  $n - j - 2$ . Clearly, there are  $n - j - 1$  such subsequences and it is easy to check that we can attach any such subsequence at the end of the sequence  $a_1 \dots a_j uv$  to obtain a 3-ascent sequence avoiding 00. If  $u$  appears in  $a$ , but  $v$  does not appear in  $a$ , then  $a$  must be of the form  $a_1 \dots a_j u \gamma$ , where  $\gamma$  is the increasing sequence  $(j+2)(j+3) \dots n$ . Similarly if  $v$  appears in  $a$ , but  $u$  does not appear in  $a$ , then  $a$  must be of the form  $a_1 \dots a_j v \gamma$ , where  $\gamma$  is the increasing sequence  $(j+2)(j+3) \dots n$ . It follows that the number of 3-ascent sequences is  $\sum_{j=2}^{n-1} \binom{j}{2} (n - j + 1)$ . One can verify by Mathematica that  $\sum_{j=2}^{n-1} \binom{j}{2} (n - j + 1) = \binom{n}{3} + \binom{n+1}{4}$ .

If  $a_j = j$ , there is one element  $u$  in  $[j]$  which does not appear in  $a_1 \dots a_j$ , so that the sequence must start out  $a_1 \dots a_j u$ . The rest of the sequence must be the increasing rearrangement of  $\{j+1, \dots, n\} - \{v\}$  for some  $v \in \{j+1, \dots, n\}$ . Thus, we have  $j-1$  choices for  $u$  and  $n-j$  choices for  $v$ . Hence the number of 3-ascent sequences  $a$  where  $\text{des}(a) = 1$  and for some  $j$ ,  $a_j > a_{j+1}$  and  $a_j = j$  is  $\sum_{j=2}^{n-1} (j-1)(n-j)$ . One can check by Mathematica that  $\sum_{j=2}^{n-1} (j-1)(n-j) = \binom{n}{3}$ .

Thus, the number of 3-ascent sequences with one descent, which avoid 00 is  $2\binom{n}{3} + \binom{n+1}{4}$ .

### Case 3 $\text{des}(a) = 2$ .

In this case, it must be that  $\max(a) = n - 1$ , so that  $a$  must contain all the elements in the sequence  $0, 1, \dots, n - 1$ . Now, suppose that the first descent of  $a$  occurs at position  $j$ . Then we have two cases depending on whether  $a_j = j$  or  $a_j = j + 1$ .

If  $a_j = j$ , there must be some  $u$ , where  $1 \leq u \leq j - 1$ , which does not appear in  $a_1 \dots a_j$  and  $a_{j+1} = u$ . We have  $j - 1$  choices for  $u$ . The sequence  $a_{j+2} \dots a_n$  must be a rearrangement of  $(j+1)(j+2) \dots (n-1)$ , which has one descent. The bottom element of the descent pair that occurs in  $a_{j+2} \dots a_n$  must equal  $s$  for some  $j+1 \leq s \leq n-2$  and the top element of the descent must equal  $t$ , where  $s+1 \leq t \leq n-1$ . It is easy to check that any choice of  $s$  and  $t$  will yield a 3-ascent sequence, so that the number of choices for the sequence  $a_{j+2} \dots a_n$  is  $\sum_{s=(j+1)}^{n-2} n - 1 - s = \binom{n-1-j}{2}$ . It follows that the number of 3-ascent sequences in this case is  $\sum_{j=2}^{n-2} (j-1) \binom{n-1-j}{2}$ , which can be shown by Mathematica to be equal to  $\binom{n-1}{4}$ .

If  $a_j = j + 1$ , then there must be two elements  $1 \leq u \leq v \leq j$  that do not appear in  $a_1 \dots a_j$ . We have  $\binom{j}{2}$  ways to choose  $u$  and  $v$ . We then have two further subcases depending on whether  $a_{j+1} = v$  or  $a_{j+1} = u$ .

If  $a_{j+1} = v$ , then our sequences start out  $a_1 \dots a_j = (j+1)v$  and where every  $u$  occurs in the sequence  $a_{j+2} \dots a_n$ , it will cause a second descent so that there are  $n - j - 1$  choices in this case. If  $a_{j+1} = u$ , then the sequence  $a_{j+2} \dots a_n$  must be a rearrangement of the sequence  $v(j+2)(j+3) \dots (n-1)$  with one descent and we can argue as we did in the case where  $a_j = j$  that there are  $\binom{n-j-1}{2}$  choices for the sequence  $a_{j+2} \dots a_n$ . Thus the total number of choices in the case where  $a_j = j + 1$  is  $\sum_{j=1}^{n-2} \binom{j}{2} \binom{n-j}{2} = \binom{n+1}{5}$  where the last equality can be checked by Mathematica.

Putting all the cases together, we see that the number of 3-ascent sequences of length  $n$ , which avoid 00 is equal to

$$\binom{n+1}{2} + 2\binom{n}{3} + \binom{n+1}{4} + \binom{n-1}{4} + \binom{n+1}{5} = \binom{n+1}{2} + 2\binom{n}{3} + \binom{n-1}{4} + \binom{n+2}{5}.$$

Thus we have the following theorem.

**Theorem 11.** For all  $n \geq 1$ ,  $a_{n,3,00} = \binom{n+1}{2} + 2\binom{n}{3} + \binom{n-1}{4} + \binom{n+2}{5}$ ,

Note that it follows from Newton's binomial theorem that

$$\begin{aligned} \sum_{n \geq 1} \binom{n+1}{2} x^n &= \frac{x}{(1-x)^3}, \quad \sum_{n \geq 1} 2\binom{n}{3} x^n = \frac{2x^3}{(1-x)^4}, \\ \sum_{n \geq 1} \binom{n-1}{4} x^n &= \frac{x^5}{(1-x)^5}, \quad \text{and} \quad \sum_{n \geq 1} \binom{n+2}{5} x^n = \frac{x^3}{(1-x)^6}. \end{aligned}$$

Adding these series together and simplifying, we have the following theorem.

**Theorem 12.**  $\sum_{n \geq 1} a_{n,3,00} x^n = \frac{x(1-3x+6x^2-5x^3+3x^4-x^5)}{(1-x)^6}$ .

We note that Burstein and Mansour [2] gave a combinatorial interpretation to the  $n$ -th element in sequence A089830 as the number of words  $w = w_1 \dots w_{n-1} \in \{1, 2, 3\}^*$ , which avoid the vincular pattern 21-2 (also denoted in the literature  $\underline{21}2$ ; see [8]). That is, there are no subsequences of the form  $w_i w_{i+1} w_j$  in  $w$  such that  $i+1 < j$  and  $w_i = w_j > w_{i+1}$ . We ask the question whether one can construct a simple bijection between such words and the set of 3-ascent sequences of length  $n$ , which avoid 00.

We note that the sequence  $(a_{n,4,00})_{n \geq 1}$  starts out 1, 4, 16, 58, 190, 564, 1526, 3794, ... This is the sequence A263851 in the OEIS [12].

**012-avoiding  $p$ -ascent sequences** Now suppose that  $a = a_1 \dots a_n$  is a  $p$ -ascent sequence such that  $a$  avoids 012. The first thing to observe is that if  $a_i = 1$  for some  $i$ , then since  $a_1 = 0$ , it must be the case that  $a_j \in \{0, 1\}$  for all  $j \geq i$ . The second thing to observe is that  $a_i \leq p$  for all  $i$ . That is, the only way that  $a$  can have an element  $a_k > p$  is if  $a_1 \dots a_{k-1}$  has at least  $a_k - p$  ascents. Since the first ascent in a  $p$ -ascent sequence must be of one of the forms 01, 02, ..., 0p, such an ascent sequence would not avoid 012.

**2-ascent sequences.** Now, suppose that  $a = a_1 \dots a_n$  is a 2-ascent sequence such that  $a$  avoids 012. If  $a$  has no 1s, then  $a_i \in \{0, 2\}$  for all  $i \geq 2$ , so that there are  $2^{n-1}$  such 2-ascent sequences. If  $a$  contains a 1, then let  $k$  be the smallest  $j$  such that  $a_j$  equals 1. It then follows that  $a_i \in \{0, 2\}$  for  $2 \leq i < k$  and  $a_j \in \{0, 1\}$  for  $k < j \leq n$ . Thus, there are  $2^{n-2}$  such 2-ascent sequences, so that the number of 2-ascent sequences that avoid 012 and contain a 1 is  $(n-1)2^{n-2}$ . Hence, for  $n \geq 1$ ,

$$a_{n,2,012} = 2^{n-1} + (n-1)2^{n-2} = (n+1)2^{n-2}. \quad (25)$$

We note that the sequence  $(a_{n,2,012})_{n \geq 1}$  starts out 1, 3, 8, 20, 48, 112, 256, ..., and this is, again, as in the case of  $(a_{n,2,10})_{n \geq 1}$ , the sequence A001792 in the OEIS [12]. Next, we will explain this fact combinatorially.

It is easy to see that each sequence counted by  $(a_{n,2,012})_{n \geq 1}$  can be obtained by taking a number of 2s (maybe none) followed by a number of 1s, and placing any number of 0s (maybe none) between these 1s and 2s making sure that the total length of the sequence is  $n$ , and this sequence begins with a 0. On the other hand, it is also straightforward to see that sequences counted by  $(a_{n,2,10})_{n \geq 1}$  are of two types: they are either of the form

$$\underbrace{0 \dots 0}_{i_0 \geq 1} \underbrace{1 \dots 1}_{i_1 \geq 1} \underbrace{2 \dots 2}_{i_2 \geq 1} \dots \underbrace{a \dots a}_{i_a \geq 1}, \quad (26)$$

where  $0, 1, \dots, a$  all appear or of the form

$$\underbrace{0 \dots 0}_{i_0 \geq 1} \underbrace{1 \dots 1}_{i_1 \geq 1} \underbrace{2 \dots 2}_{i_2 \geq 1} \dots \underbrace{a \dots a}_{i_a \geq 1} \underbrace{(a+2) \dots (a+2)}_{i_{a+2} \geq 1} \underbrace{(a+3) \dots (a+3)}_{i_{a+3} \geq 1} \underbrace{(a+4) \dots (a+4)}_{i_{a+4} \geq 1} \dots, \quad (27)$$

where  $a \geq 0$  exists. A bijection between the classes of sequences is given by turning sequences of the form (26) into

$$\underbrace{0 \dots 0}_{i_0} \underbrace{2 0 \dots 0}_{i_1-1} \underbrace{2 0 \dots 0}_{i_2-1} \dots \underbrace{2 0 \dots 0}_{i_a-1},$$

and the sequences of the form (27) into

$$\underbrace{0 \dots 0}_{i_0} \underbrace{2 0 \dots 0}_{i_1-1} \underbrace{2 0 \dots 0}_{i_2-1} \dots \underbrace{2 0 \dots 0}_{i_a-1} \underbrace{1 0 \dots 0}_{i_{a+2}-1} \underbrace{1 0 \dots 0}_{i_{a+3}-1} \underbrace{1 0 \dots 0}_{i_{a+4}-1} \dots$$

**3-ascent sequences.** Now, suppose that  $a = a_1 \dots a_n$  is a 3-ascent sequence such that  $a$  avoids 012. If  $a$  has no 1s, then  $a_i \in \{0, 2, 3\}$  for all  $i \geq 2$ . It is then easy to see that if  $b_1 \dots b_n$  is the sequence that arises from  $a_1 \dots a_n$  by replacing each 2 by a 1 and each 3 by a 2, then  $b$  is a 2-ascent sequence that avoids 012. Thus, there are  $(n+1)2^{n-2}$  such sequences. Now, suppose that  $a$  contains a 1. Then let  $k$  be the smallest  $j$  such that  $a_j$  equals 1. It then follows that  $a_i \in \{0, 2, 3\}$  for  $2 \leq i < k$  and  $a_j \in \{0, 1\}$  for  $k < j \leq n$ . It is then easy to see that if  $b_1 \dots b_{k-1}$  is the sequence that arises from  $a_1 \dots a_{k-1}$  by replacing each 2 by a 1 and each 3 by a 2, then  $b_1 \dots b_{k-1}$  is a 2-ascent sequence that avoids 012. Thus, from our argument above, it follows that there are  $k2^{k-3}$  choices for  $a_1 \dots a_{k-1}$  and  $2^{n-k}$  choices for  $a_{k+1} \dots a_n$ . Therefore, given  $k$ , we have  $k2^{n-3}$  choices for  $a$ . Thus,

$$a_{n,3,012} = (n+1)2^{n-2} + \sum_{k=2}^n k2^{n-3} = 2^{n-4}(n^2 + 5n + 2) \quad (28)$$

where the last equality can be checked by Mathematica. We note that the sequence  $(a_{n,3,012})_{n \geq 1}$  starts out 1, 4, 13, 38, 104, 272, 688, ... and this is the sequence A049611 in the OEIS [12] having several combinatorial interpretations.

**$p$ -ascent sequences for an arbitrary  $p$ .** In general, we can obtain a simple recursion for  $a_{n,p,012}$ . That is, suppose that  $a = (a_1, \dots, a_n)$  is a  $p$ -ascent sequence such that  $a$  avoids 012. Now, if  $a$  has no 1s, then  $a_i \in \{0, 2, 3, \dots, p\}$  for all  $i \geq 2$ . It is then easy to see that if  $b = (b_1, \dots, b_n)$  is the sequence that arises from  $a$  by replacing each  $i \geq 2$ , by an  $i-1$ , then  $b$  is a  $(p-1)$ -ascent sequences that avoids 012. Thus, there are  $a_{n,p-1,012}$  such sequences. Now suppose that  $a$  contains a 1. Then let  $k$  be the smallest  $j$  such that  $a_j$  equals 1. It then follows that  $a_i \in \{0, 2, 3, \dots, p\}$  for  $2 \leq i < k$  and  $a_j \in \{0, 1\}$  for  $k < j \leq n$ . It is then easy to see that if  $b_1 \dots b_{k-1}$  is the sequence that arises from  $a_1 \dots a_{k-1}$  by replacing each  $i \geq 2$  by an  $i-1$ , then  $b_1 \dots b_{k-1}$  is a 2-ascent sequences that avoids 012. It follows that there are  $a_{k-1,p-1,012}$  choices for  $a_1 \dots a_{k-1}$  and  $2^{n-k}$  choices for  $a_{k+1} \dots a_n$ . Thus, given  $k$ , we have  $2^{n-k}a_{k-1,p-1,012}$  choices for  $a$ . It follows that

$$a_{n,p,012} = a_{n,p-1,012} + \sum_{k=2}^n a_{k-1,p-1,012}2^{n-k}. \quad (29)$$

For example, using our formula for  $a_{n,3,012}$ , one can compute that  $a_{n,4,012} = \frac{2^{n-5}}{3}(n^3 + 12n^2 + 29n + 6)$ . The sequence  $(a_{n,4,012})_{n \geq 1}$  starts out 1, 5, 19, 63, 192, 552, 1520, 4048, 10496, 26264, ... and this is the sequence A049612 in the OEIS [12].

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